·Review·

Statistical virtual eye model based on wavefront aberration

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Foundation items: National Natural Science Foundation of China (No. 61173182, No. 61179071); Applied Basic Research Project (No. 2011JY0124) and International Cooperation and Exchange Project of Sichuan Province (No. 2012HH0004)

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Received: 2012-4-29 Accepted:2012-9-18

Abstract

• Wavefront aberration affects the quality of retinal image directly. This paper reviews the representation and reconstruction of wavefront aberration, as well as the construction of virtual eye model based on Zernike polynomial coefficients. In addition, the promising prospect of virtual eye model is emphasized.

• KEYWORDS: wavefront aberration; wavefront reconstruction; virtual eye; Zernike polynomial

DOI:10.3980/j.issn.2222-3959.2012.05.15

Wang JM, Liu CL, Luo YN, Liu YG, Hu BJ. Statistical virtual eye model based on wavefront aberration. *Int J Ophthalmol* 2012;5(5):620-624

INTRODUCTION

The refractive system of human eye is mainly composed of cornea, lens, vitreous and others. However, just like other optical instruments, the human eye is not an ideal optical system. The difference of wavefront between the ideal optical system and the actual measured optical system is wavefront aberration [1]. Besides the mainly traditional low-order aberration [2], high-order aberration also exists in human eye. Low-order aberrations include tilts, defocus and astigmatism. While, other optical defects of refractive system, such as spherical aberration, coma, are included in high-order aberration. Refractive surgery can modify the human eye's optical system so as to improve the visual quality of the retina, which mostly corrects low-order aberrations such as myopia, hyperopia and astigmatism. It can improve the patients' eyesight significantly, but some problems, such as poor night vision, glare monocular diplopia, which are mainly caused by high-order aberration, remain unsolved [3]. In order to improve the visual quality, it is necessary to further research the wavefront aberration. This paper reviews the representation and reconstruction of wavefront aberration, as well as methods of developing virtual eye from wavefront aberration. In addition, the promising prospect of virtual eye model is emphasized.

REPRESENTATION OF WAVEFRONT ABERRATION

The methods for representing wavefront aberration of human eye are as a list or a bar chart showing coefficients of normalized Zernike polynomials [4], and in the form of a topographical map.

Coefficients of Zernike Polynomials Porter *et al* [5] found that Zernike polynomials can describe the wavefront aberration of human eye effectively in 2001. Smolek and Klyce *et al* [6] claimed that the higher of the order used in the Zernike polynomials, the greater of the fitting error will be. And Zernike method fails to capture all high-order aberration for eyes after corneal surgery which was concluded by Klyce[7].

Typically, the ophthalmic coordinate system is used to describe the Zernike polynomials as shown in Figure 1. Zernike polynomials are orthogonal over the unit circle, so the coefficients are independent to each other [8]. The coefficients of Zernike polynomials can be formulated in double index or single index.

Double index representation The Zernike polynomials are usually presented as z_n^m (ρ,θ) in polar coordinate system, with the index n describing the order of the aberration and the index m representing the azimuthal frequency of the sinusoidal. The radius parameter is designed by the letter ρ

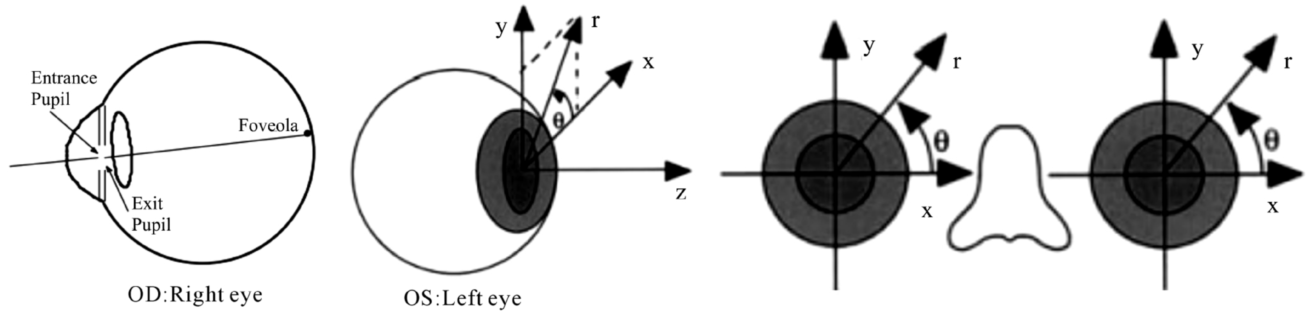

Figure 1 Ophthalmic coordinate system.

continuous over its range of 0 to 1.0, and θ means the azimuthal component continuous over its range of 0 to 2π. The wavefront aberration function $W(x,y)$ can by presented as followings:

$$W(x,y)=\sum_{n,m} c_n^m z_n^m \tag{1}$$

$$z_n^m(\rho,\theta)=\begin{cases} N_n^m R_n^{|m|}(\rho)\cos m\theta & m>0 \\ -N_n^m R_n^{|m|}(\rho)\sin m\theta & m<0 \end{cases} \tag{2}$$

$$R_n^{|m|}(\rho)=\sum_{s=0}^{(n-|m|)/2}\frac{(-1)^s(n-s)!}{s![ns(n+|m|-s)]![0.5*(n-|m|-s)]!}\rho^{n-2s}$$
$$N_n^m=\sqrt{\frac{2(n+1)}{1+\delta_{m,0}}}\ \text{Or}\ N_n^m=\sqrt{(2-\delta_{m,0})(n+1)} \tag{3}$$

Where, if $m=0$, $\delta_{m,0}=1$, otherwise, $\delta_{m,0}=0$.

Zernike polynomials are orthogonal in the sense that they satisfy the following equation:

$$\int_0^1 \rho d_\rho \int_0^{2\Pi} z_n^m z_{n'}^{m'} d_\theta = \Pi\delta_{n,n'}\delta_{m,m'}, \tag{4}$$

Whter, if, $n=n'$, $\delta_{n,n'}=1$, otherwise, $\delta_{n,n'}=0$

If, $m=m'$, $\delta_{m,m'}=1$, otherwise, $\delta_{m,m'}=0$.

Single index representation In the above representation, the Zernike mode coefficient depends on two parameters, n and m. Occasionally, a standard single indexing scheme is more arbitrary for describing the coefficients of Zernike polynomials with the index j. While this scheme only be used for bar plots for the coefficients. Their relationships are as follows[9]:

$$Z_j = Z_n^m$$
$$j=\frac{n(n+2)+m}{2}$$
$$n=\left\lceil\frac{-3+\sqrt{9+8j}}{2}\right\rceil$$
$$m=2j-n(n+2) \tag{5}$$

Topographical Map Wavefront aberration is normally given in form of series of polynomial coefficients, root mean square (RMS) values, 2D contour plots, 3D surface plots, or vergence map. The vergence map is the standard format for topographic representation of wavefront aberration of human eye, and it can be obtained from the wavefront senor directly, without the wavefront aberration reconstruction[10]. Map is convenient for doctors to catch the distribution of the patient's eye aberration with the intuitionistic representation provided by wavefront aberration diagram.

WAVEFRONT ABERRATION RECONSTRUCTION

The representation of wavefront aberration of human eye is introduced above. In general, aberrometry measures the slope of wavefront in the eye [10], and we can reconstruct the wavefront aberration of human eye using Zernike reconstruction algorithm by the slope [11,12]. Jeong *et al*[13] reconstructed wavefront aberration by use of Zernike polynomials in 2007. The simulated wavefront data from corneal topography to calculate corneal aberration was obtained by Geunyong Yoon *et al* [14], and they concluded that the Zernike method outperformed the Fourier method using the slop information.

The Theory of Wavefront Aberration Reconstruction In the wavefront aberration reconstruction, to posit and track the pupil automatically, the initial parameters should be set firstly and faculae image can be gained by Hartmann-Shark sensor. Then the centroid of every faculae was calculated as the actual focus of wavefront. Finally, coefficients of Zernike polynomials, which present wavefront aberration, are calculated by a series of formulas.

Centroid Localization The reconstruction of wavefront aberration based on the distance between standard center and actual center. So it is very important to select an accurate standard value, especially when the difference is very large [15-17]. Gravity method can be used to calculate the actual center in wavefront aberration reconstruction. We can calculate the centroid according to [11,14,18,19,20]:

$$x_i=\frac{\sum_k^{m_i} I_{i,j}x_{i,j}}{\sum_k^{m_i} I_{i,j}} \qquad y_i=\frac{\sum_k^{m_i} I_{i,j}y_{i,j}}{\sum_k^{m_i} I_{i,j}} \tag{6}$$

Where $I_{i,j}$ represents the amplitude of the pixel in the sub-aperture, $x_{i,j}$ and $y_{i,j}$ are the coordinates of j th pixel in the i th sub-aperture, respectively.

Zernike polynomials are orthogonal over the unit circle in theory, and hence the discrete sampling data is required to be normalized before using Zernike polynomials for wavefront aberration reconstruction. (While there is a necessary normalize the discrete sampling data first.) These data is up to the center and the radius, the normalized coordinate can be obtained as follows: $x'=(x-x_0)/r_0, y'=(y-y_0)/r_0$. Where (x,y) is the initial coordinate, (x',y') means the normalized coordinate, the center coordinate is (x_0,y_0), the symbol of r_0 represents the radius. And the error of (x_0,y_0) and r_0 will affect the accuracy.

The same size of pupil is absolutely necessary when comparing wavefront aberration, because the size of pupil will affect Zernike coefficients directly, Wang *et al*[21] showed that the increase of pupil's diameter would not lead to all high order aberration increase correspondingly. Spherical aberration from 4mm to 5mm pupil size increased less than from 5mm to 6mm, and other high order aberration showed only a small increase with pupil dilation[21].

Methods of Wavefront Aberration Reconstruction

Least Square Method In 2009, ANSI presented that Zernike coefficients can be generated based on the principle of least squares using the slope information of wavefront. This method is fitting the data to a set of polynomials $\{Z_j\}$, where j ranges from unity to the total number of polynomials[22],and the polynomials are Zernike polynomials.

$$Zc=G \tag{7}$$

Where Z contains slope information for Zernike polynomials, the vector of Zernike coefficients is, every column of G can be obtained from wavefront aberrometer. The aim of reconstruction is to determine the values of the coefficients in the vector^c. The solution of least squares method is given by

$$c=[ZTZ]^{-1}Z^TG \tag{8}$$

Where, Z^T is transpose of matrix Z, and $[Z^TZ]^{-1}$ is the inverse matrix of Z^TZ.

With, the final reconstruction of wavefront aberration is:

$$W(x,y)=\sum_{k=1}^{j} c_k z_k(x,y) \tag{9}$$

The method of calculating Zernike coefficients by Least Square Method is relatively stable in the case of noise, which is noted by J. Y. in 1980. He also showed that the result of Gram-Schmidt orthogonal transform method is competitive [23]. Besides, the equation is often seriously ill and contradictory when the data is larger with n>>j, where n is the number of rows of Z, and the exact solution can not be obtained. In 2006, Cao *et al* [24] obtained fitting coefficients by the way of making the contradiction equation Orthogonal and triangular through Householder transformation. The result showed that relative error of each order coefficient is less than 10%[24].

Gram –Schmidt Orthogonal Transformation Instead of using the in version method as least square method, which is numerically unstable, Gram-Schmidt Orthogonal Transformation method doesn't determine the Zernike coefficients directly. And the method seeks to generate an orthogonal set p from matrix.

$$G=pb \tag{10}$$

Following the Gram-Schmidt orthogonal Transform procedure, the Zernike polynomials can be presented in terms of orthogonal sets P_k up to the order j; i.e.,

$$Z=[p_1,p_2,\cdots,p_j]\begin{bmatrix}1 & e_{12} & \cdots & e_{1j}\\ 0 & 1 & \cdots & e_{2j}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix} \tag{11}$$

The coefficients of Zernike polynomials are determined by the following relations:

$$\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_j\end{bmatrix}=\begin{bmatrix}1 & e_{12} & \cdots & e_{1j}\\ 0 & 1 & \vdots & e_{2j}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_j\end{bmatrix} \tag{12}$$

And then the expression of ^{c} is:

$$\begin{cases} c_j=b_j \\ c_i=b_i-\sum_{k=i+1}^{j} e_{ik}c_k \quad i=1,2,\cdots,j-1\end{cases} \tag{13}$$

Gram-Schmidt Orthogonal Transformation improves the status of equation to a certain degree, but it still can not get rid of the construction of equation.

Householder Transformation The least square problem by Householder transformation is to make the coefficient matrix of orthogonal and triangular by Householder transformation.

$$Q\bullet Z=\begin{pmatrix}R\\ 0\end{pmatrix}, \tag{14}$$

Where R represents upper triangular nonsingular real matrix with j order, 0 is zero matrix with n-j order, Q means orthogonal matrix with n order.

$$QgG=\begin{pmatrix}A\\ B\end{pmatrix}, \tag{15}$$

Where,A is j-dimensional column vector,B is n-j-dimensional column vector.

$$Q\delta=Q(G-Zc)=\begin{pmatrix}A\\ B\end{pmatrix}-\begin{pmatrix}R\\ 0\end{pmatrix}c=\begin{pmatrix}A-Rc\\ B\end{pmatrix}, \tag{16}$$

Where, $\|\delta\|_2^2=\|Q\delta\|_2^2=\|A-Rc\|_2^2+\|B\|_2^2$, $A-Rc$ is vanished, $\|\delta\|_2^2$ will get the minimum and $Rc=A$, $c=R^{-1}A$.

Householder transformation is very stable since it can avoid the construction of equation. In 2009, Chen *et al* [24] made a conclusion that the Householder transformation was better than Gram-Schmidt orthogonal transformation and Singular Value Decomposition by the standard of time-consuming and reconstruction precision. However, Gao *et al* [12] thought this algorithm's calculation was too large in 2010.

Singular Value Decomposition (SVD) The Singular Value Decomposition algorithm is used to obtain generalized inverse of the coefficient matrix. The theory is simple and the calculation steps are few. Especially, it is

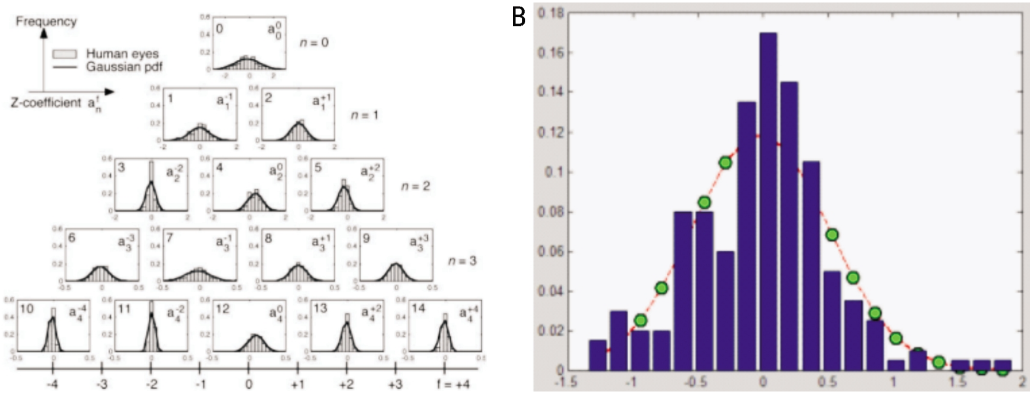

Figure 2 Frequency histogram of the Zernike polynomial coefficients a_n^f is the Zernike polynomial coefficients, n is the order, f is the frequency, the bar is frequency of the Zernike coefficients, the solid line is the Gauss fitting curve. The probability distribution of the Zernike coefficients in line was distributed with a Gaussian distribution as Figure 2B.

easy to program. In 2010, Gao *et al* [12] claimed that the singular value decomposition algorithm was the ideal wavefront aberration reconstruction algorithm. We can obtain wavefront aberration by wavefront aberration reconstruction based on the above methods.

VIRTUAL EYES

Physiological parameters (wavefront aberration) in the traditional model of the eye are fixed, but the wavefront aberration is individual. Virtual eye model with the individual aberration of wavefront is more close to real eyes. Recently, there exists two ways to generate virtual eyes, Gaussian model simulation and Monte Carlo simulation.

Gaussian Model Simulation As to functional virtual eyes based on Zernike coefficients, a group of Zernike coefficients could represent a virtual eye in Thibos's theory. According to the experimental results of the wavefront aberration, they confirmed that the coefficients of wavefront aberration polynomials are consistent with Gaussian model (Figure 2). They measure 100 optometry students in the age range 22-35 years and get 200 groups data from health eyes for a 6mm pupil.

Gaussian model generates 1000 groups random number (1000 group coefficients) by computer. In 2009, by the comparison of human eyes and statistical virtual eyes, it was claimed by them that the statistical virtual eyes slightly exaggerated the decline in retinal image quality [26-29].

For the population of normal eyes given in Thibos's study, each individual Zernike coefficient was reasonably well described as a random variable of Gaussian model, which has an appealing feature of simplicity. But for some abnormal eyes, this model is possibly inaccurate, it is necessary for us to verify it and find a more general and accurate model.

Monte Carlo Simulation In 2009, Li *et al* [30] obtained 20 groups Zernike coefficients and formed a 27 ×20 matrix. They got a group of Zernike coefficients as virtual eyes by selecting one item from each row randomly. Based on this method, they got 100 virtual eyes by using Mat-lab.

The method of Monte Carlo simulation selected for generating virtual eyes is very simple and useful. Monte Carlo method is a random and reliable method in mathematics, but whether the wavefront aberration of human eye, which has its own characteristics, can be generated by that may be a little lack of theoretical basis.

APPLICATION PROSPECT OF VIRTUAL EYE

Wavefront aberration, presented by Zernike polynomials, has a wide range of applications in eye research and clinic [31].While the researches between high order wavefront aberration and vision are still in primary stage.

We may quantify the quality of the retinal image by the point-spread function (PSF), optical transfer function (OTF), or the wavefront aberration function. The building of statistical virtual eyes makes it possible to simulate the influence of wavefront aberration to retinal image and to study individual variation in retinal image quality. Therefore, the establishment and application of statistical virtual eye model have a great deal of potential applications. One application of the model is to explore the human visual physiological mechanisms and computer vision theory; other potential applications are carrying out individual corneal refractive surgery, designing functional intraocular lens and corneal contact lens, evaluating of visual function, or exploiting ophthalmological apparatus and astronomical telescope with high image quality, and so on. Besides, that statistical virtual eye model is capable of generate large experimental reference data for various kinds of researches by computer. One advantage of this approach is that we can improve research efficiency by saving cost and time. Another advantage is that the model is generated as a stochastic process rather than a system with fixed parameters.

Researchers have made some works in the statistical virtual eye model. But, it is still a problem to make the statistical model of the virtual eye closer to the real eye. Simulate the eye in normal or various pathological states and assess the visual quality of different surgical approach or different intraocular lens implantation are unrealized, too. These problems still need a lot of hard works.

Acknowledgements: The authors thank Zeng-Xi Huang for critical comments and help with the paper revising.

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